Int J Theor Phys (2008) 47: 44–60
DOI 10.1007/s10773-007-9477-0

Quantum Computational Logics and Possible Applications

Maria Luisa Dalla Chiara · Roberto Giuntini · Roberto Leporini · Giuliano Toraldo di Francia

Received: 19 December 2006 / Accepted: 28 June 2007 / Published online: 28 July 2007

Abstract In quantum computational logics *meanings* of formulas are identified with quantum information quantities: systems of *qubits* or, more generally, *mixtures* of systems of qubits. We consider two kinds of quantum computational semantics: (1) a *compositional* semantics, where the *meaning* of a compound formula is determined by the meanings of its parts; (2) a *holistic* semantics, which makes essential use of the characteristic "holistic" features of the quantum-theoretic formalism. The compositional and the holistic semantics turn out to characterize the same logic. In this framework, one can introduce the notion of *quantum-classical truth table*, which corresponds to the most natural way for a quantum computer to calculate classical tautologies.

Quantum computational logics can be applied to investigate different kinds of semantic phenomena where *holistic*, *contextual* and *gestaltic* patterns play an essential role (from natural languages to musical compositions).

Keywords Quantum computation · Quantum logic

M.L.D. Chiara
Dipartimento di Filosofia, Università di Firenze, Via Bolognese 52, 50139 Firenze, Italy
e-mail: dallachiara@unifi.it

R. Giuntini
Dipartimento di Scienze Pedagogiche e Filosofiche, Università di Cagliari, Via Is Mirrionis 1, 09123 Cagliari, Italy
e-mail: giuntini@unica.it

R. Leporini (✉)
Dipartimento di Matematica, Statistica, Informatica e Applicazioni, Università di Bergamo, Via dei Caniana 2, 24127 Bergamo, Italy
e-mail: roberto.leporini@unibg.it

G.T. di Francia
Dipartimento di Fisica, Università di Firenze, Polo Scientifico, Sesto Fiorentino, 50100 Firenze, Italy

1 Introduction

Quantum computational logics are new forms of quantum logic, that arise as a natural logical abstraction from the theory of quantum logical gates in quantum computation. In the standard semantics of these logics, formulas denote quantum information quantities (systems of qubits, or, more generally, mixtures of systems of qubits), while the logical connectives are interpreted as logical operations defined in terms of special quantum logical gates.

Let us first sum up some basic notions of quantum computation. Consider the two-dimensional Hilbert space $\mathbb{C}^2$, where any vector $|\psi\rangle$ is represented by a pair of complex numbers. Let $\mathcal{B}^{(1)} = \{|0\rangle, |1\rangle\}$ be the *canonical orthonormal basis* for $\mathbb{C}^2$ such that $|0\rangle = (0, 1)$; $|1\rangle = (1, 0)$.

Definition 1.1 (Qubit) A *qubit* is a unit vector $|\psi\rangle$ of the space $\mathbb{C}^2$.

Hence, any qubit has the following form:

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \text{ (where } a_0, a_1 \in \mathbb{C} \text{ and } |a_0|^2 + |a_1|^2 = 1).$$

From an intuitive point of view, a qubit can be regarded as a quantum variant of the classical notion of bit: a kind of "quantum perhaps". In this framework, the two basis-elements $|0\rangle$ and $|1\rangle$ represent the two classical bits 0 and 1, respectively. From a physical point of view, a qubit represents a *state* of a single particle, carrying an atomic piece of quantum information. In order to carry the information stocked by n qubits, we need of course a compound system, consisting of n particles.

Definition 1.2 (Quregister) An n-qubit system (also called *n-quregister*) is a unit vector in the n-fold tensor product Hilbert space $\otimes^n\mathbb{C}^2 := \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n\text{-times}}$ (where $\otimes^1\mathbb{C}^2 := \mathbb{C}^2$).

We will use $x, y, \ldots$ as variables ranging over the set $\{0, 1\}$. At the same time, $|x\rangle, |y\rangle, \ldots$ will range over the basis $\mathcal{B}^{(1)}$. Any factorized unit vector $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ of the space $\otimes^n\mathbb{C}^2$ will be called a *classical register*. We will also write $|x_1, \ldots, x_n\rangle$ instead of $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$. The set $\mathcal{B}^{(n)}$ of all classical registers is an orthonormal basis for the space $\otimes^n\mathbb{C}^2$.

Quregisters are *pure states*: maximal pieces of information about the particles under consideration. Both in quantum theory and in quantum information, one cannot help referring also to *mixed states* (or *mixtures*), which represent pieces of information that are not maximal and might be enriched [8]. In the framework of quantum computation, mixed states (mathematically represented by density operators of an appropriate Hilbert space) are also called *qumixes*.

Definition 1.3 (Qumix) A *qumix* is a density operator of $\otimes^n\mathbb{C}^2$ (where $n \geq 1$).

Needless to say, quregisters correspond to particular qumixes that are *pure states* (i.e. projections onto one-dimensional closed subspaces of a given $\otimes^n\mathbb{C}^n$). We will indicate by $\mathcal{Q}(\otimes^n\mathbb{C}^2)$ the set of all qureguisters of $\otimes^n\mathbb{C}^2$, while $\mathfrak{D}(\otimes^n\mathbb{C}^2)$ will denote the set of all density operators of $\otimes^n\mathbb{C}^2$. Hence the two sets $\mathcal{Q} = \bigcup_{n=1}^{\infty} \mathcal{Q}(\otimes^n\mathbb{C}^2)$ and $\mathfrak{D} = \bigcup_{n=1}^{\infty} \mathfrak{D}(\otimes^n\mathbb{C}^2)$ will represent the set of all possible quregisters and the set of all possible qumixes, respectively.

For semantic aims, it is useful to distinguish the *true* from the *false* registers in any space $\otimes^n\mathbb{C}^2$.

Definition 1.4 (True and false registers) Let $|x_1, \ldots, x_n\rangle$ be a register of $\otimes^n \mathbb{C}^2$.

- $|x_1, \ldots, x_n\rangle$ is called *true* iff $x_n = 1$;
- $|x_1, \ldots, x_n\rangle$ is called *false* iff $x_n = 0$.

The idea is that any classical register corresponds to a classical truth-value that is determined by its last element. Hence, in particular, the bit $|1\rangle$ corresponds to the truth-value *Truth*, while the bit $|0\rangle$ corresponds to the truth-value *Falsity*.

On this basis, we can identify, in any space $\otimes^n \mathbb{C}^2$, two special projection-operators ($P_1^{(n)}$ and $P_0^{(n)}$) that represent, in this framework, the *Truth-property* and the *Falsity-property*, respectively. The projection $P_1^{(n)}$ is determined by the closed subspace spanned by the set of all true registers, while $P_0^{(n)}$ is determined by the closed subspace spanned by the set of all false registers. As is well known, in quantum theory, projections have the role of *mathematical representatives* of possible physical properties of the quantum objects under investigation. Hence, it turns out that *Truth* and *Falsity* behave here as special cases of physical properties.

As a consequence, one can naturally apply the *Born rule* that determines *the probability-value that a quantum system in a given state satisfies a given property*. Consider any qumix ρ, which represents a possible state of a quantum system in the space $\otimes^n \mathbb{C}^2$. By applying the Born rule, we obtain that the probability-value that a physical system in state ρ satisfies the *Truth-property* $P_1^{(n)}$ is the number $\mathtt{tr}(P_1^{(n)}\rho)$ (where $\mathtt{tr}$ is the *trace functional*). This suggests the following natural definition of the notion of *probability* of a given qumix.

Definition 1.5 (Probability of a qumix) For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$, $\mathtt{p}(\rho) := \mathtt{tr}(P_1^{(n)}\rho)$.

From an intuitive point of view, $\mathtt{p}(\rho)$ represents the probability that the information stocked by the qumix ρ is true. In the particular case where ρ corresponds to the qubit $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$, we obtain that $\mathtt{p}(\rho) = |a_1|^2$.

Given a quregister $|\psi\rangle$, we will also write $\mathtt{p}(|\psi\rangle)$ instead of $\mathtt{p}(P_{|\psi\rangle})$, where $P_{|\psi\rangle}$ is the density operator represented by the projection onto the one-dimensional subspace spanned by the vector $|\psi\rangle$.

An interesting relation connects qumixes with the real numbers in the interval [0, 1]. Any real number $\lambda \in [0, 1]$ uniquely determines a qumix $\rho_\lambda^{(n)}$ (for any $n \in \mathbb{N}^+$):

$$\rho_\lambda^{(n)} := (1-\lambda)k_n P_0^{(n)} + \lambda k_n P_1^{(n)}$$

(where k_n is a normalization coefficient). From an intuitive point of view, $\rho_\lambda^{(n)}$ represents a *mixture of pieces of information* that might correspond to the *Truth* with probability λ.

In quantum computation information is processed by *quantum logical gates* (briefly, *gates*): unitary operators that transform quregisters into quregisters. Being unitary, gates represent characteristic *reversible logical operations*. The canonical gates (which are studied in the literature) can be naturally generalized to qumixes. We will consider here the following gates: the *negation*, the *Petri–Toffoli gate* and the *square root of the negation*.

Let us first describe these gates in the framework of quregisters.

Definition 1.6 (The negation) For any $n \geq 1$, the negation on $\otimes^n \mathbb{C}^2$ is the linear operator $\mathtt{Not}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the basis $\mathcal{B}^{(n)}$:

$$\mathtt{Not}^{(n)}(|x_1, \ldots, x_n\rangle) := |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In other words, $\mathtt{Not}^{(n)}$ inverts the value of the last element of any basis-vector of $\otimes^n \mathbb{C}^2$.

Definition 1.7 (The Petri–Toffoli gate) For any $m \geq 1$ and any $n \geq 1$ the Petri–Toffoli gate is the linear operator $\mathtt{T}^{(m,n,1)}$ defined on $\otimes^{m+n+1}\mathbb{C}^2$ such that for every element $|x_1,\dots,x_m\rangle \otimes |y_1,\dots,y_n\rangle \otimes |z\rangle$ of the basis $\mathcal{B}^{(m+n+1)}$:

$$\mathtt{T}^{(m,n,1)}(|x_1,\dots,x_m\rangle \otimes |y_1,\dots,y_n\rangle \otimes |z\rangle) := |x_1,\dots,x_m\rangle \otimes |y_1,\dots,y_n\rangle \otimes |x_m y_n \boxplus z\rangle,$$

where $\boxplus$ represents the sum modulo 2.

One can easily show that both $\mathtt{Not}^{(n)}$ and $\mathtt{T}^{(m,n,1)}$ are unitary operators.

Consider now the set $\mathcal{Q}$ of all possible quregisters. The gates $\mathtt{Not}$ and $\mathtt{T}$ can be uniformly defined on this set in the expected way:

$$\mathtt{Not}(|\psi\rangle) := \mathtt{Not}^{(n)}(|\psi\rangle), \quad \text{if } |\psi\rangle \in \otimes^n\mathbb{C}^2,$$

$$\mathtt{T}(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle) := \mathtt{T}^{(m,n,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle),$$
$$\text{if } |\psi\rangle \in \otimes^m\mathbb{C}^2,\ |\varphi\rangle \in \otimes^n\mathbb{C}^2 \text{ and } |\chi\rangle \in \mathbb{C}^2.$$

On this basis, a conjunction $\mathtt{And}$, a disjunction $\mathtt{Or}$, can be defined for any pair of quregisters $|\psi\rangle$ and $|\varphi\rangle$:

$$\mathtt{And}(|\psi\rangle, |\varphi\rangle) := \mathtt{T}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle);$$

$$\mathtt{Or}(|\psi\rangle, |\varphi\rangle) := \mathtt{Not}(\mathtt{And}(\mathtt{Not}(|\psi\rangle), \mathtt{Not}(|\varphi\rangle))).$$

Clearly, $|0\rangle$ represents an "ancilla" in the definition of $\mathtt{And}$.

The gates we have considered so far are, in a sense, "semiclassical". A quantum logical behavior only emerges in the case where our gates are applied to superpositions. When restricted to classical registers, such operators turn out to behave as classical (reversible) truth-functions. We will now consider an important example of a *genuine quantum gate* that transforms classical registers (elements of $\mathcal{B}^{(n)}$) into quregisters that are superpositions. This gate is the *square root of the negation*.

Definition 1.8 (The square root of the negation) For any $n \geq 1$, the square root of the negation on $\otimes^n\mathbb{C}^2$ is the linear operator $\sqrt{\mathtt{Not}}^{(n)}$ such that for every element $|x_1,\dots,x_n\rangle$ of the basis $\mathcal{B}^{(n)}$:

$$\sqrt{\mathtt{Not}}^{(n)}(|x_1,\dots,x_n\rangle) := |x_1,\dots,x_{n-1}\rangle \otimes \frac{1}{2}((1+i)|x_n\rangle + (1-i)|1-x_n\rangle),$$

where $i := \sqrt{-1}$.

One can easily show that $\sqrt{\mathtt{Not}}^{(n)}$ is a unitary operator. The basic property of $\sqrt{\mathtt{Not}}^{(n)}$ is the following:

$$\text{for any } |\psi\rangle \in \otimes^n\mathbb{C}^2, \quad \sqrt{\mathtt{Not}}^{(n)}(\sqrt{\mathtt{Not}}^{(n)}(|\psi\rangle)) = \mathtt{Not}^{(n)}(|\psi\rangle).$$

In other words, applying twice the square root of the negation means negating.

From a logical point of view, $\sqrt{\mathtt{Not}}^{(n)}$ can be regarded as a "tentative partial negation" (a kind of "half negation") that transforms *precise pieces of information* into *maximally uncertain* ones. For, we have:

$$\mathtt{p}(\sqrt{\mathtt{Not}}^{(1)}(|1\rangle)) = \frac{1}{2} = \mathtt{p}(\sqrt{\mathtt{Not}}^{(1)}(|0\rangle)).$$

As expected, also $\sqrt{\mathtt{Not}}$ can be uniformly defined on the set $\mathfrak{R}$ of all quregisters.

Interestingly enough, the gate $\sqrt{\mathtt{Not}}$ seems to represent a typically *quantum logical operation* that does not admit any counterpart either in classical logic or in standard fuzzy logics (see [4]).

The gates considered so far can be naturally generalized to qumixes [8]. When our gates will be applied to density operators, we will write: $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$, $\mathbb{T}$, $\mathtt{AND}$, $\mathtt{OR}$ (instead of $\mathtt{Not}$, $\sqrt{\mathtt{Not}}$, $\mathtt{T}$, $\mathtt{And}$, $\mathtt{Or}$).

Definition 1.9 (The negation) For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$,

$$\mathtt{NOT}^{(n)}(\rho) := \mathtt{Not}^{(n)} \rho \mathtt{Not}^{(n)}.$$

Definition 1.10 (The square root of the negation) For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$,

$$\sqrt{\mathtt{NOT}}^{(n)}(\rho) := \sqrt{\mathtt{Not}}^{(n)} \rho \sqrt{\mathtt{Not}}^{(n)*},$$

where $\sqrt{\mathtt{Not}}^{(n)*}$ is the adjoint of $\sqrt{\mathtt{Not}}^{(n)}$.

It is easy to see that for any $n \in \mathbb{N}^+$, both $\mathtt{NOT}^{(n)}(\rho)$ and $\sqrt{\mathtt{NOT}}^{(n)}(\rho)$ are qumixes of $\mathfrak{D}(\otimes^n \mathbb{C}^2)$.

Definition 1.11 (The conjunction) Let $\rho \in \mathfrak{D}(\otimes^m \mathbb{C}^2)$ and $\sigma \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$.

$$\mathtt{AND}^{(m,n,1)}(\rho, \sigma) = \mathbb{T}^{(m,n,1)}(\rho, \sigma, P_0^{(1)}) := \mathtt{T}^{(m,n,1)}(\rho \otimes \sigma \otimes P_0^{(1)}) \mathtt{T}^{(m,n,1)}.$$

Like in the quregister-case, the gates $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$, $\mathbb{T}$, $\mathtt{AND}$, $\mathtt{OR}$ can be uniformly defined on the set $\mathfrak{D}$ of all qumixes.

An interesting preorder relation can be defined on the set $\mathfrak{D}$ of all qumixes.

Definition 1.12 (Preorder) $\rho \preceq \sigma$ iff the following conditions hold:

(i) $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$;
(ii) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sigma)) \leq \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$.

One immediately shows that $\preceq$ is reflexive and transitive, but not antisymmetric (counterexamples can be easily found in $\mathfrak{D}(\mathbb{C}^2)$).

An equivalence relation can be then defined on $\mathfrak{D}$:

Definition 1.13 $\sigma \equiv \tau$ iff $\sigma \preceq \tau$ and $\tau \preceq \sigma$.

One can prove that $\equiv$ is a congruence relation with respect to the operations $\mathtt{AND}$, $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$.

2 Quantum Trees

We consider here a *minimal quantum computational language* $\mathcal{L}$ that contains a privileged atomic formula **f** (whose intended interpretation is the *Falsity*) and the following primitive connectives: the *negation* ($\neg$), the *square root of the negation* ($\sqrt{\neg}$), a ternary *conjunction*

$\bigwedge$ (which corresponds to the Petri–Toffoli gate). For any formulas α and β, the expression $\bigwedge(\alpha, \beta, \mathbf{f})$ is a formula of $\mathcal{L}$. In this framework, the usual conjunction $\alpha \wedge \beta$ is dealt with as metalinguistic abbreviation for the ternary conjunction $\bigwedge(\alpha, \beta, \mathbf{f})$. We will use the following metavariables: $\mathbf{q}, \mathbf{r}, \ldots$ for atomic formulas and $\alpha, \beta, \ldots$ for formulas. The connective disjunction ($\vee$) is supposed to be defined via the *de Morgan law* ($\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$).

Any formula α of $\mathcal{L}$ describes a *quantum circuit* that can be applied to an input, represented by a qumix living in a Hilbert space whose dimension depends on the linguistic form of α. Let us first introduce some useful syntactical notions. By *atomic complexity* of a formula α (indicated by $At(\alpha)$) we mean the number of occurrences of atomic formulas in α. For instance, $At(\neg \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) = 3$. Since the atomic complexity of α determines the dimension of the Hilbert space where a qumix representing information about α should live, the space $\otimes^{At(\alpha)}\mathbb{C}^2$ will be also called the *semantic space* of α. We will briefly write $\mathcal{H}^\alpha$, instead of $\otimes^{At(\alpha)}\mathbb{C}^2$.

Any formula α can be naturally decomposed into its parts, giving rise to a special configuration called the *syntactical tree* of α (indicated by $STree^\alpha$).

Roughly, $STree^\alpha$ can be represented as a sequence of *levels*:

$$\begin{gathered} Level_k(\alpha) \\ \vdots \\ Level_1(\alpha), \end{gathered}$$

where:

- each $Level_i(\alpha)$ (with $1 \leq i \leq k$) is a sequence of subformulas of α;
- the *bottom level* ($Level_1(\alpha)$) consists of α;
- the *top level* ($Level_k(\alpha)$) is the sequence of all atomic occurrences in α;
- for any i (with $1 \leq i < k$), $Level_{i+1}(\alpha)$ is the sequence obtained by dropping the *principal connective* in all molecular formulas occurring at $Level_i(\alpha)$, and by repeating all the atomic formulas that possibly occur at $Level_i(\alpha)$.

As an example, consider the following formula: $\alpha = \mathbf{q} \wedge \neg\mathbf{q} = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$. The syntactical tree of α is the following configuration:

$$\begin{aligned} Level_3(\alpha) &= (\mathbf{q}, \mathbf{q}, \mathbf{f}); \\ Level_2(\alpha) &= (\mathbf{q}, \neg\mathbf{q}, \mathbf{f}); \\ Level_1(\alpha) &= \Big(\bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})\Big). \end{aligned}$$

By *Height* of α (indicated by $Height(\alpha)$) we mean the number of levels of the syntactical tree of α. For instance, $Height(\bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) = 3$.

The syntactical tree of α (which represents a purely syntactical object) uniquely determines a sequence of gates that are all defined on the semantic space of α. We will call this gate-sequence the *qubit tree* of α. Consider a formula α such that $At(\alpha) = t$ and $Height(\alpha) = k$. Let $Level_i^j(\alpha)$ represent the j-th *node* of $Level_i(\alpha)$. Each $Level_i^j(\alpha)$ (where $1 \leq i < Height(\alpha)$) can be naturally associated to a unitary operator Op_i^j, according to the following *operator-rule*:

$$Op_i^j := \begin{cases} I^{(1)} & \text{if } Level_i^j(\alpha) \text{ is an atomic formula;} \\ \mathtt{Not}^{(r)} & \text{if } Level_i^j(\alpha) = \neg\beta \text{ and } At(\beta) = r; \\ \sqrt{\mathtt{Not}}^{(r)} & \text{if } Level_i^j(\alpha) = \sqrt{\neg}\beta \text{ and } At(\beta) = r; \\ \mathtt{T}^{(r,s,1)} & \text{if } Level_i^j(\alpha) = \bigwedge(\beta, \gamma, \mathbf{f}), At(\beta) = r \text{ and } At(\gamma) = s, \end{cases}$$

where $I^{(1)}$ is the identity operator of $\mathbb{C}^2$.

On this basis, one can associate a gate G_i^α to each $Level_i(\alpha)$ (such that $1 \le i < Height(\alpha)$):

$$G_i^\alpha := \bigotimes_{j=1}^{|Level_i(\alpha)|} Op_i^j,$$

where $|Level_i(\alpha)|$ is the length of the sequence $Level_i(\alpha)$.

Being the tensor product of unitary operators, every G_i^α turns out to be a unitary operator. One can easily show that all G_i^α are defined on the same space, $\mathcal{H}^\alpha$.

Definition 2.1 (The qubit tree of α) The *qubit tree* of α (denoted by $QTree^\alpha$) is the gate-sequence

$$(G_1^\alpha, \ldots, G_{Height(\alpha)-1}^\alpha)$$

that is uniquely determined by the syntactical tree of α.

As an example, consider again the formula: $\alpha = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$. The qubit tree of α is represented by the gate-sequence (G_1^α, G_2^α), where:

$$G_1^\alpha = \mathtt{T}^{(1,1,1)};$$

$$G_2^\alpha = I^{(1)} \otimes \mathtt{Not}^{(1)} \otimes I^{(1)}.$$

As we have seen, qubit trees consist of unitary operators (which can be applied to quregisters). The notion of qubit tree can be naturally generalized to qumixes. In such a case we will speak of *qumix trees*, and we will call *quantum tree* either a qubit tree or a qumix tree. Let $(G_1^\alpha, \ldots, G_{k-1}^\alpha)$ be the qubit tree of α. We can define the following sequence of functions on the set $\mathfrak{D}(\mathcal{H}^\alpha)$:

$$^{\mathfrak{D}}G_1^\alpha(\rho) = G_1^\alpha \rho G_1^{\alpha *}$$

$$\vdots$$

$$^{\mathfrak{D}}G_{k-1}^\alpha(\rho) = G_{k-1}^\alpha \rho G_{k-1}^{\alpha *}.$$

One can easily prove that, for any $\rho \in \mathfrak{D}(\mathcal{H}^\alpha)$ and for any i $(1 \le i \le k-1)$, $^{\mathfrak{D}}G_i^\alpha(\rho)$ is a density operator of $\mathfrak{D}(\mathcal{H}^\alpha)$. The sequence

$$QumTree^\alpha = (^{\mathfrak{D}}G_1^\alpha, \ldots, ^{\mathfrak{D}}G_{k-1}^\alpha)$$

will be called the *qumix tree* of α, while the elements of a qumix tree will be called *qumix gates*. Apparently, all qumix gates are bijections. Hence, qumix trees (as well as qubit trees) represent reversible information processes.

Consider now a formula α and let $({}^{\mathfrak{D}}G_1^{\alpha}, \ldots, {}^{\mathfrak{D}}G_{k-1}^{\alpha})$ be the qumix tree of α. Any choice of a qumix ρ in $\mathcal{H}^{\alpha}$ determines a sequence $(\rho_k, \ldots, \rho_1)$ of qumixes of $\mathcal{H}^{\alpha}$, where:

$$\begin{gathered}\rho_k = \rho, \\ \rho_{k-1} = {}^{\mathfrak{D}}G_{k-1}^{\alpha}(\rho_k), \\ \vdots \\ \rho_1 = {}^{\mathfrak{D}}G_1^{\alpha}(\rho_2).\end{gathered}$$

The qumix ρ_k can be regarded as a possible *input-information* concerning the atomic parts of α, while ρ_1 represents the *output-information* about α, given the input-information ρ_k. Each ρ_i corresponds to the *information* about $Level_i(\alpha)$, given the input-information ρ_k. Any sequence of this kind will be called α*-computation* (with input ρ_k and with output ρ_1). Alternatively, we can also refer to the qubit tree $(G_1^{\alpha}, \ldots, G_{k-1}^{\alpha})$ of α and define, in the expected way, the notion of *pure* α*-computation* (with input $|\psi_k\rangle$ and with output $|\psi_1\rangle$).

How to determine an information about the parts of α under a given input? It is natural to apply the standard quantum-theoretic rule that determines the *states of the parts of a compound system.*

Suppose that:

$$Level_i(\alpha) = \beta_{i_1}, \ldots, \beta_{i_r}.$$

We have:

$$\mathcal{H}^{\alpha} = \mathcal{H}^{\beta_{i_1}} \otimes \ldots \otimes \mathcal{H}^{\beta_{i_r}}.$$

We know that $QumTree^{\alpha}$ and the choice of an input ρ_k (in $\mathcal{H}^{\alpha}$) determine a sequence of qumixes:

$$\begin{gathered}\rho_k \leftrightsquigarrow Level_k(\alpha) = (\mathbf{q}_1, \ldots, \mathbf{q}_t), \\ \vdots \\ \rho_i \leftrightsquigarrow Level_i(\alpha) = (\beta_{i_1}, \ldots, \beta_{i_r}), \\ \vdots \\ \rho_1 \leftrightsquigarrow Level_1(\alpha) = (\alpha).\end{gathered}$$

Consider $red^j(\rho_i)$, the *reduced state of* ρ_i *with respect to the* j*-th subsystem.*[1] From a semantic point of view, this state can be regarded as a *contextual information* about β_{i_j} (the subformula of α occurring at the j-th position at $Level_i(\alpha)$) under the input ρ_k. Apparently, a contextual information about a subformula is generally a mixture.

[1]We recall that $red^j(\rho_i)$ is the unique density operator that satisfies the following condition: for any self-adjoint operator A^j of $\mathcal{H}^{\beta_j}$,

$$\mathtt{tr}(red^j(\rho_i)A^j) = \mathtt{tr}(\rho_i(I^1 \otimes \ldots \otimes I^{j-1} \otimes A^j \otimes I^{j+1} \otimes \ldots \otimes I^r)),$$

(where I^h is the identity operator of $\mathcal{H}^{\beta_h}$). As a consequence, ρ_i and $red^j(\rho_i)$ are statistically equivalent with respect to the j-th subsystem of the compound system described by ρ_i.

An interesting situation arises when the qumix ρ_k, representing a global information about the atomic parts of α, is an *entangled* pure state.[2]

As an example, consider the formula $\alpha = \neg \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$ (which represents an example of the *noncontradiction principle* formalized in the quantum computational language). The input-information might be the following entangled state:

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|000\rangle \rightsquigarrow Level_4(\alpha) = (\mathbf{q}, \mathbf{q}, \mathbf{f}).$$

The reduced states of $|\psi_4\rangle$ turn out to be the following:

$$red^1\left(\frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|000\rangle\right) = \frac{1}{2}P_0^{(1)} + \frac{1}{2}P_1^{(1)} = red^2\left(\frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|000\rangle\right)$$
$$red^3\left(\frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|000\rangle\right) = P_0^{(1)}.$$

Hence, the contextual information about both occurrences of $\mathbf{q}$ is the (proper) mixture $\frac{1}{2}P_0^{(1)} + \frac{1}{2}P_1^{(1)}$. At the same time, the contextual information about $\mathbf{f}$ is projection $P_0^{(1)}$ (representing the *Falsity*).

Quantum trees can be naturally regarded as examples of quantum circuits that compute outputs under given inputs. Since both qubit trees and qumix trees are determined by the syntactical tree of a given formula, one can also say that any formula α of the quantum computational language plays the role of an intuitive and "economical" description of a quantum circuit, called α-*quantum circuit*.

3 Compositional and Holistic Quantum Computational Semantics

Two kinds of quantum computational semantics have been investigated: a *compositional* and a *holistic* semantics. In the *compositional* semantics, the meaning of a molecular formula is determined by the meanings of its parts (like in classical logic). In this framework, the input-information about the top level of the syntactical tree of a formula α is always associated to a factorized state $\rho_1 \otimes \ldots \otimes \rho_t$, where t is the atomic complexity of α and $\rho_1, \ldots, \rho_t$ are qumixes of $\mathbb{C}^2$. As a consequence, the meaning of a molecular α cannot be a pure state, if the meanings of some atomic parts of α are proper mixtures.

The *holistic quantum compositional semantics*[3] is based on a more "liberal" assumption: the input information about the top-level of the syntactical tree of α can be represented by any qumix "living" in the semantic space of α. As a consequence, the meanings of all levels of $STree^\alpha$ are not, generally, factorized states.

Suppose that:

$$Level_i(\alpha) = (\beta_1, \ldots, \beta_r).$$

[2] As is well known, the basic features of an *entangled state* $|\psi\rangle$ are the following: (1) $|\psi\rangle$ is a maximal information (a pure state) that describes a compound physical system S; (2) the pieces of information determined by $|\psi\rangle$ about the parts of S are, generally, non-maximal (proper mixtures). Hence, the information about the *whole* is more precise than the information about the *parts*.

[3] See [5]; in [4] we have presented a weaker version of the holistic semantics.

As we have seen, the space $\mathcal{H}^\alpha$ can be naturally regarded as the Hilbert space of a compound physical system consisting of r parts (mathematically represented by the spaces $\mathcal{H}^{\beta_1}, \ldots, \mathcal{H}^{\beta_r}$), where each part may be compound. On this basis, for any qumix ρ_i (associated to $Level_i(\alpha)$) and for any node $Level_i^j(\alpha)$, we can consider the *reduced state* $red^j(\rho_i)$ with respect to the j-th subsystem of the system described by ρ_i. From an intuitive point of view, $red^j(\rho_i)$ describes the j-th subsystem on the basis of the *global* information ρ_i. Since $Level_i(\alpha) = (\beta_1, \ldots, \beta_r)$, the qumix $red^j(\rho_i)$ (which is a density operator of the space $\mathcal{H}^{\beta_j}$) represents a *possible meaning* of the sentence β_j.

We can now introduce the basic definitions of the holistic semantics. The main concept is the notion of *holistic quantum computational model*: a function $\mathtt{Hol}$ that assigns to any formula α of the quantum computational language a *global meaning*, which cannot be generally inferred from the meanings of the parts of α. Of course (like in the standard semantic approaches), the function $\mathtt{Hol}$ shall respect the logical form of α.

In order to define the concept of *holistic quantum computational model*, we will first introduce the notions of *atomic holistic model* and of *tree holistic model.*

Definition 3.1 (Atomic holistic model) An *atomic holistic model* is a map $\mathtt{Hol}^{At}$ that associates a qumix to any formula α of $\mathcal{L}$, satisfying the following conditions:

(1) $\mathtt{Hol}^{At}(\alpha) \in \mathfrak{D}(\mathcal{H}^\alpha)$;
(2) Let $At(\alpha) = n$ and $Level_{Heigth(\alpha)} = \mathbf{q}_1, \ldots, \mathbf{q}_n$. Then,
  (2.1) if $\mathbf{q}_j = \mathbf{f}$, then $red^j(\mathtt{Hol}^{At}(\alpha)) = P_0$;
  (2.2) if $\mathbf{q}_j$ and $\mathbf{q}_h$ are two occurrences in α of the same atomic formula, then $red^j(\mathtt{Hol}^{At}(\alpha)) = red^h(\mathtt{Hol}^{At}(\alpha))$.

Apparently, $\mathtt{Hol}^{At}(\alpha)$ represents a *global interpretation* of the atomic formulas occurring in α. At the same time, $red^j(\mathtt{Hol}^{At}(\alpha))$, the *reduced state* of the compound system (described by $\mathtt{Hol}^{At}(\alpha)$) with respect to the j-th subsystem, represents a *contextual meaning* of $\mathbf{q}_j$ with respect to the *global meaning* $\mathtt{Hol}^{At}(\alpha)$. Conditions (2.1) and (2.2) guarantee that $\mathtt{Hol}^{At}(\alpha)$ is well behaved. For, the contextual meaning of $\mathbf{f}$ is always the *Falsity*, while two different occurrences (in α) of the same atomic formula have the same contextual meaning.

The map $\mathtt{Hol}^{At}$ (which assigns a meaning to the top-level of the syntactical tree of any sentence α) can be naturally extended to a map $\mathtt{Hol}^{Tree}$ that assigns a meaning to each level of the syntactical tree of any α, following the prescriptions of the qumix tree of α.

Consider a formula α such that:

$$QumTree^\alpha = ({}^{\mathfrak{D}}G_1^\alpha, \ldots, {}^{\mathfrak{D}}G^\alpha_{Heigth(\alpha)-1}).$$

The map $\mathtt{Hol}^{Tree}$ is defined as follows:

$$\mathtt{Hol}^{Tree}(Level_{Heigth(\alpha)}) = \mathtt{Hol}^{At}(\alpha),$$
$$\mathtt{Hol}^{Tree}(Level_i(\alpha)) = {}^{\mathfrak{D}}G_i^\alpha(\mathtt{Hol}^{Tree}(Level_{i+1}(\alpha))$$

(where $Heigth(\alpha) > i \geq 1$).

On this basis, one can naturally define the notion of *holistic* (*quantum computational*) *model* of $\mathcal{L}$.

Definition 3.2 (Holistic model) A map $\mathtt{Hol}$ that assigns to any formula α a qumix of the space $\mathcal{H}^\alpha$ is called a *holistic (quantum computational) model* of $\mathcal{L}$ iff there exists an atomic holistic model $\mathtt{Hol}^{At}$ s.t.:

$$\mathtt{Hol}(\alpha) = \mathtt{Hol}^{\mathit{Tree}}(\mathit{Level}_1(\alpha)),$$

where $\mathtt{Hol}^{\mathit{Tree}}$ is the extension of $\mathtt{Hol}^{At}$.

Given a formula γ, $\mathtt{Hol}$ determines the *contextual meaning*, with respect to the context $\mathtt{Hol}(\gamma)$, of any occurrence of a subformula β in γ.

Definition 3.3 (Contextual meaning of a node) Let β be a subformula of γ occurring at the j-th position of the i-th level of the syntactical tree of γ. We indicate by $\beta[^i_j]$ the node of STree^γ corresponding to such occurrence. The contextual meaning of $\beta[^i_j]$ with respect to the context $\mathtt{Hol}(\gamma)$ is defined as follows:

$$\mathtt{Hol}^\gamma(\beta[^i_j]) = \mathit{red}^j(\mathtt{Hol}^{\mathit{Tree}}(\mathit{Level}_i(\gamma))).$$

Hence, we have:

$$\mathtt{Hol}^\gamma(\gamma) = \mathtt{Hol}^{\mathit{Tree}}(\mathit{Level}_1(\gamma)) = \mathtt{Hol}(\gamma).$$

Suppose that $\beta[^i_j]$ and $\beta[^h_k]$ are two nodes of the syntactical tree of γ, representing two occurrences of the same subformula β. One can show that:

$$\mathtt{Hol}^\gamma(\beta[^i_j]) = \mathtt{Hol}^\gamma(\beta[^h_k]).$$

In other words, two different occurrences of one and the same subformula in a formula γ receive the same contextual meaning with respect to the context $\mathtt{Hol}(\gamma)$.

On this basis, one can define the *contextual meaning* of a subformula β of γ, with respect to the context $\mathtt{Hol}(\gamma)$:

$$\mathtt{Hol}^\gamma(\beta) := \mathtt{Hol}^\gamma(\beta[^i_j]),$$

where $\beta[^i_j]$ is any occurrence of β at a node of STree^γ.

Suppose now that β is a subformula of two different formulas γ and δ. Generally, we have:

$$\mathtt{Hol}^\gamma(\beta) \neq \mathtt{Hol}^\delta(\beta).$$

In other words, formulas may receive different contextual meanings in different contexts!

Apparently, $\mathtt{Hol}^\gamma$ is a partial function that only assigns meanings to the subformulas of γ. Given a formula γ, we will call the partial function $\mathtt{Hol}^\gamma$ a *contextual holistic model* of the language.

In this framework, compositional models can be described as limit-cases of holistic models.

Definition 3.4 (Compositional model) A model $\mathtt{Hol}$ is called *compositional* iff the following condition is satisfied for any formula α: $\mathtt{Hol}^{At}(\alpha) = \mathtt{Hol}(\mathbf{q}_1) \otimes \ldots \otimes \mathtt{Hol}(\mathbf{q}_t)$, where $\mathbf{q_1}, \ldots, \mathbf{q_t}$ are the atomic formulas occurring in α.

As expected, unlike holistic models, compositional models are context-independent. Suppose that β is a subformula of two different formulas γ and δ. We have:

$$\mathtt{Hol}^\gamma(\beta) = \mathtt{Hol}^\delta(\beta) = \mathtt{Hol}(\beta).$$

The notion of logical consequence in the framework of the holistic quantum computational semantics represents a reasonable variant of the standard notions of logical consequence.

Let us first define the notion of *consequence in a given contextual model*.

Definition 3.5 (Consequence in a given contextual model $\mathtt{Hol}^\gamma$) A formula β is a consequence of a formula α in a given contextual model $\mathtt{Hol}^\gamma$ ($\alpha \models_{\mathtt{Hol}^\gamma} \beta$) iff

1. α and β are subformulas of γ;
2. $\mathtt{Hol}^\gamma(\alpha) \preceq \mathtt{Hol}^\gamma(\beta)$ (where $\preceq$ is the preorder relation defined in 1.12).

Definition 3.6 (Logical consequence (in the holistic semantics)) A formula β is a consequence of a formula α (in the holistic semantics) iff for any formula γ such that α and β are subformulas of γ and for any $\mathtt{Hol}$,

$$\alpha \models_{\mathtt{Hol}^\gamma} \beta.$$

We call **HQCL** the logic that is semantically characterized by the logical consequence relation we have just defined. Hence, $\alpha \models_{\mathbf{HQCL}} \beta$ iff for any formula γ such that α and β are subformulas of γ and for any $\mathtt{Hol}$,

$$\alpha \models_{\mathtt{Hol}^\gamma} \beta.$$

At the same time, by *compositional quantum computational logic* (**CQCL**) we mean the logic that is semantically characterized by the class of all compositional quantum computational models. Hence, $\alpha \models_{\mathbf{CQCL}} \beta$ iff for any compositional model $\mathtt{Hol}$,

$$\alpha \models_{\mathtt{Hol}} \beta.$$

Although the basic ideas of the holistic and of the compositional quantum computational semantics are quite different, one can prove that **HQCL** and **CQCL** are the same logic (see [5]). In other words, for any formulas α and β,

$$\alpha \models_{\mathbf{HQCL}} \beta \quad \text{iff} \quad \alpha \models_{\mathbf{CQCL}} \beta.$$

This means that the logics (formalized in our “poor” sentential languages) are not able to capture the difference between an analytical and a holistic semantic procedure.

Since **HQCL= CQCL**, we will simply speak of *quantum computational logic* (denoted by **QCL**). One is dealing with a nonstandard form of *unsharp quantum logic*, where the noncontradiction principle breaks down ($\not\models_{\mathbf{QCL}} \neg(\alpha \wedge \neg\alpha)$), while conjunction is not idempotent ($\alpha \not\models_{\mathbf{QCL}} \alpha \wedge \alpha$). Interestingly enough, distributivity is here violated “in the wrong direction” with respect to orthodox quantum logic. For, $\alpha \wedge (\beta \vee \gamma) \models_{\mathbf{QCL}} (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, but not the other way around!

4 A Quantum Computational Description of Classical Truth-Tables

The quantum computational semantics (mainly in its holistic version) seems to represent a new theory of meanings that might find interesting applications also to other fields, quite far from microphysics.

We will first consider a logical application: a natural synthetic description of *classical truth-tables*. Consider the classical sublanguage $\mathcal{L}^C$ of $\mathcal{L}$, whose only connectives are the classical (reversible) connectives $\neg$ and $\bigwedge$. Let α be a formula of $\mathcal{L}^C$ with atomic occurrences $\mathbf{q}_1, \ldots, \mathbf{q}_t$.

Definition 4.1 A classical register $|x_1, \ldots, x_t\rangle$ is called a *classical truth-value assignment* for α iff there is an atomic holistic model $\mathtt{Hol}^{At}$ of the language $\mathcal{L}^C$ s.t.

$$\mathtt{Hol}^{At}(\alpha) = |x_1, \ldots, x_t\rangle.$$

Apparently, any classical truth-value assignment for α represents a possible (classical) interpretation of the top level of the syntactical tree of α (the sequence of the atomic formulas occurring in α).

Let Val^α be the set of all possible classical truth-value assignments for α and let n^α be its cardinal number.

Definition 4.2 A quregister $|\psi\rangle$ of $\mathcal{H}^\alpha$ is called a *classical truth-value assignment system* for α iff

$$|\psi\rangle = \sum \left\{ \frac{1}{\sqrt{n^\alpha}} |x_1, \ldots, x_t\rangle : |x_1, \ldots, x_t\rangle \in Val^\alpha \right\}.$$

Clearly, any formula α has a unique classical truth-value assignment system (indicated by $|\psi_C^\alpha\rangle$). Unlike classical truth-value assignments (which are classical registers), the classical truth-value assignment system for α is a genuine superposition, where all amplitudes are the same number ($\frac{1}{\sqrt{n^\alpha}}$).

On this basis, we can naturally define the notion of *quantum-classical truth table* of α.

Definition 4.3 (Quantum-classical truth table) The *quantum-classical truth table* of α is the pure α-computation with input $|\psi_C^\alpha\rangle$.

Suppose that $QubTree^\alpha = (G_{k-1}^\alpha, \ldots, G_1^\alpha)$. Then, the *quantum-classical truth table* of α is the following sequence of quregisters living in the semantic space of α:

$$\begin{aligned} |\psi_k\rangle &= |\psi_C^\alpha\rangle, \\ |\psi_{k-1}\rangle &= G_{k-1}^\alpha(|\psi_k\rangle), \\ &\vdots \\ |\psi_1\rangle &= G_1^\alpha(|\psi_2\rangle). \end{aligned}$$

In this framework, the notion of *classical tautology* can be then defined in the expected way.

Definition 4.4 (Classical tautology) A formula α is a *classical tautology* iff $\mathtt{p}(|\psi_1\rangle) = 1$, where $|\psi_1\rangle$ is the output of the quantum-classical truth table $(|\psi_k\rangle, \ldots, |\psi_1\rangle)$ of α.

Example 4.1 The noncontradiction law $\alpha = \neg \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$ (which is not generally valid in quantum computational logics) is a quantum-classical tautology. We have: $Val^\alpha =$

$\{|110\rangle, |000\rangle\}$, and $n^\alpha = 2$. The quantum-classical truth table of α is the following sequence of quregisters:

$$|\psi_4\rangle = |\psi_C^\alpha\rangle = \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|000\rangle,$$

$$|\psi_3\rangle = (I^{(1)} \otimes \mathtt{NOT}^{(1)} \otimes I^{(1)})(|\psi_4\rangle) = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{\sqrt{2}}|010\rangle,$$

$$|\psi_2\rangle = \mathtt{T}^{(1,1,1)}(|\psi_3\rangle) = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{\sqrt{2}}|010\rangle,$$

$$|\psi_1\rangle = \mathtt{NOT}^{(3)}(|\psi_2\rangle) = \frac{1}{\sqrt{2}}|101\rangle + \frac{1}{\sqrt{2}}|011\rangle,$$

whence, $\mathtt{p}(|\psi_1\rangle) = 1$.

Interestingly enough, our definition of *quantum-classical truth table* could not be given in the framework of the compositional semantics. For, by definition, the classical truth-value assignment system $|\psi_C^\alpha\rangle$ is not a factorized state.

Using quantum-classical truth tables seems to represent the most natural way for a quantum computer to calculate classical tautologies. The quantum-classical truth table of a classical formula is here described as the "time-evolution" of a pure state (living in the semantic space of our formula), while the probability of being true is calculated as a quantum probability. One could say that such a reconstruction permits us to justify the classical logical world by means of "quantum computational eyes".

5 Looking for Abstract Holistic Structures

To what extent can the holistic version of quantum computational semantics provide a mathematical formalism for a theory of meanings, where holistic and contextual features play a relevant role?

As is well known, human perception like thinking seems to be essentially *synthetic*. We never perceive an object by *scanning* it point by point. We instead form right away a *Gestalt*, i.e. a global idea of it. *Rational activity*, as well, seems to be essentially based on *gestaltic patterns*. As an example, we can refer to the chess-game. Since the total number of possible games is finite, the games could be divided into three sets: (1) the set of all games won by white. (2) the set of all games won by black. (3) the set of all games ending in a draw. Why could not a fantastic computer of the white player always choose from the first set?

A strong human player certainly must perform some rapid calculations, but above all he must be able first to perceive a *Gestalt* of the position, and then to assess by experience the probabilities of its different issues. Some time ago, the famous computer Deep Blue has beaten the world champion Gary Kasparov. Soon after the game, Kasparov is said to have protested because he suspected "something human" having taken place in the strategy of Deep Blue. Did Kasparov have a hunch of any "gestaltic phenomenon"?

Gestalt-thinking cannot be adequately represented in the framework of classical semantics, which is basically *analytical* and *compositional*: the meaning of a *compound* expression is always determined by the meanings of its *parts*. At the same time, meanings are *non-ambiguous* and *sharp*.

All this renders classical semantics hardly applicable to an adequate analysis of natural languages and of artistic contexts, where holistic and ambiguous features seem to play a relevant role. As a significant example, we often mention the final verse of the poem *L'Infinito* by Giacomo Leopardi:

E 'l naufragar m'è dolce in questo mare
(And drowning in this sea is sweet to me).

Here, the poetic result seems to be essentially connected with the following semantic relation: the meanings of the component expressions "naufragar" (drowning), "dolce" (sweet), "mare" (sea) do not correspond here to the most common meanings. By the way, there is no sea in Recanati, Leopardi's native village which the poem refers to. However the usual meanings of our expressions are somehow present and ambiguously correlated with the metaphorical meanings that are evoked by the whole poem. Needless to say, this represents a quite typical semantic situation in poetry. Also musical compositions are concerned with *meanings* that are intrinsically holistic, contextual and ambiguous.

In the holistic quantum computational semantics the following conditions hold:

(1) *global meanings* (which may correspond to a *Gestalt*) are essentially *vague*, because they leave semantically undecided many relevant properties of the objects under investigation;
(2) any global meaning determines some *partial meanings*, which are generally vaguer than the global one;
(3) meanings (*Gestalten*) can be generally represented as *superpositions* of meanings, possibly associated to probability-values;
(4) meanings are dealt with as intrinsically dynamic objects.

In spite of its appealing features, the present version of the quantum computational semantics is strongly "Hilbert-space dependent". This certainly represents a shortcoming for all applications, where real and complex numbers do not generally play any significant role (as happens, for instance, in the case of natural and of artistic languages).

Is it sensible to look for an *abstract quantum computational semantics*? We sketch here only a first basic step in this direction: the definition of *abstract quregister structure*. In this framework, abstract quregisters are identified with some special objects (not necessarily living in a Hilbert space), while gates are reversible functions that transform quregisters into quregisters. In order to stress the relation between abstract and concrete quregister-structures, we will use the familiar ket-notation also for abstract quregisters. From an intuitive point of view, abstract quregisters represent pieces of information that are generally *uncertain*, while (abstract) registers are special examples of quregisters that stock a *certain* information. Any (abstract) quregister is associated to a given length n, and lives in a subdomain $\mathcal{Q}^{(n)}$ of the domain $\mathcal{Q}$ of all possible quregisters. The preorder relation $\preceq$ is here primitive and has the following intuitive interpretation: $|\psi\rangle \preceq |\varphi\rangle$ iff the information encoded by $|\varphi\rangle$ is "closer to the truth" than the information encoded by $|\psi\rangle$. Another primitive relation, called *quconsistency*, permits us to define an abstract notion of *superposition*.

Definition 5.1 (Abstract quregister structure) An abstract quregister structure is a system

$$\langle \mathcal{Q}, \clubsuit, \preceq, \mathtt{Not}, \sqrt{\mathtt{Not}}, \mathtt{T}, |0\rangle, |1\rangle \rangle,$$

where the following conditions hold:

(1) $\mathcal{Q}$ is the set of all *abstract quregisters* (briefly, *quregisters*), indicated by $|\psi\rangle, |\varphi\rangle, \ldots$.

   (a) $\mathcal{Q} = \bigcup_{n\geq 1} \mathcal{Q}^{(n)}$, where $\mathcal{Q}^{(n)}$ is the set of all *quregisters of length n*, indicated by $|\psi\rangle^{(n)}, |\varphi\rangle^{(n)}, \dots$.
   (b) The Cartesian product $\mathcal{Q}^{(m)} \times \mathcal{Q}^{(n)}$ is embeddable into $\mathcal{Q}^{(m+n)}$. We indicate by $|\psi\rangle^{(m)} \otimes |\varphi\rangle^{(n)}$ the element of $\mathcal{Q}^{(m+n)}$ that corresponds to the pair $(|\psi\rangle^{(m)}, |\varphi\rangle^{(n)})$. We have: $|\psi\rangle^{(m)} \otimes (|\varphi\rangle^{(n)} \otimes |\chi\rangle^{(p)}) = (|\psi\rangle^{(m)} \otimes |\varphi\rangle^{(n)}) \otimes |\chi\rangle^{(p)}$.
(2) For any $n \geq 1$, $\mathfrak{R}^{(n)}$ is the set of all *registers of length n*. The elements of $\mathfrak{R}^{(n)}$ are represented as sequences $|x_1, \dots, x_n\rangle$, where $x_i \in \{0, 1\}$. The set $\mathfrak{R}^{(1)} = \{|0\rangle, |1\rangle\}$ is called the set of the two *abstract bits*.
   (a) $\mathfrak{R}^{(n)} \subseteq \mathcal{Q}^{(n)}$;
   (b) $\mathfrak{R}^{(m+n)}$ is in one-to-one correspondence with the Cartesian product $\mathfrak{R}^{(m)} \times \mathfrak{R}^{(n)}$. We indicate by $|x_1, \dots, x_m, y_1, \dots, y_n\rangle$ the register in $\mathfrak{R}^{(m+n)}$ that corresponds to the pair $(|x_1, \dots, x_m\rangle, |y_1, \dots, y_n\rangle)$.
(3) For any $n \geq 1$, $\clubsuit$ is a map that associates to n a binary reflexive and symmetric relation $\clubsuit^n$ (called *quconsistency*) that may hold between quregisters of length n.
   (a) $|x_1, \dots, x_n\rangle \clubsuit^n |y_1, \dots, y_n\rangle \curvearrowright |x_1, \dots, x_n\rangle = |y_1, \dots, y_n\rangle$;
   (b) any quregister of length n is quconsistent with at least one register of length n. Let $Reg(|\psi\rangle^{(n)}) = \{|x_1, \dots x_n\rangle : |x_1, \dots x_n\rangle \clubsuit^n |\psi\rangle^{(n)}\}$. We say that $|\psi\rangle^n$ is a *superposition* of the elements of $Reg(|\psi\rangle^{(n)})$.
(4) $\preceq$ is a *preorder relation* on $\mathcal{Q}$. This permits one to define the following equivalence relation:

$$|\psi\rangle \equiv |\varphi\rangle := |\psi\rangle \preceq |\varphi\rangle \quad \text{and} \quad |\varphi\rangle \preceq |\psi\rangle.$$

(5) Registers satisfy the following conditions:
   (a) $|x_1, \dots, x_m\rangle \preceq |y_1, \dots, y_n, 1\rangle$;
   (b) $|x_1, \dots, x_m, 0\rangle \preceq |y_1, \dots, y_n\rangle$.
(6) If $|1\rangle \preceq |x_1, \dots, x_n\rangle$ for any register $|x_1, \dots, x_n\rangle \in Reg(|\psi\rangle^{(n)})$, then $|1\rangle \preceq |\psi\rangle^{(n)}$.
(7) $\texttt{Not}$, $\sqrt{\texttt{Not}}$, $\texttt{T}$ are maps that assume as values *abstract logical gates* (briefly, *gates*). By *gate on* $\mathcal{Q}^{(n)}$ we mean a map $G^{(n)}$ that satisfies the following conditions:
   (a) $G^{(n)}$ is a bijection on $\mathcal{Q}^{(n)}$;
   (b) $|x_1, \dots, x_n\rangle \clubsuit^n |\psi\rangle^{(n)} \curvearrowright G^{(n)}(|x_1, \dots, x_n\rangle) \clubsuit^n G^{(n)}(|\psi\rangle^{(n)})$;
   (c) $\equiv$ is a congruence with respect to $G^{(n)}$.
   Condition (a) guarantees that gates are reversible logical operations, while (b) represents an abstract linearity requirement.
(8) For any $n \geq 1$, $\texttt{Not}$ associates to n the gate $\texttt{Not}^{(n)}$ (defined on $\mathcal{Q}^{(n)}$) that satisfies the following conditions:
   (a) $\texttt{Not}^{(n)}(|x_1, \dots, x_n\rangle) \equiv |x_1, \dots, x_{n-1}, 1 - x_n\rangle$;
   (b) $\texttt{Not}^{(n)}(\texttt{Not}^{(n)}(|\psi\rangle^{(n)})) \equiv |\psi\rangle^{(n)}$.
(9) For any $n \geq 1$, $\sqrt{\texttt{Not}}$ associates to n the gate $\sqrt{\texttt{Not}}^{(n)}$ (defined on $\mathcal{Q}^{(n)}$) that satisfies the following conditions:
   (a) $\sqrt{\texttt{Not}}^{(n)}(|x_1, \dots, x_n\rangle) \equiv |x_1, \dots, x_{n-1}\rangle \otimes \sqrt{\texttt{Not}}^{(1)}(|x_n\rangle)$;
   (b) $\sqrt{\texttt{Not}}^{(n)}(\sqrt{\texttt{Not}}^{(n)}(|\psi\rangle^{(n)})) \equiv \texttt{Not}^{(n)}(|\psi\rangle^{(n)})$.
   (c) $\sqrt{\texttt{Not}}^{(n)}(|x_1, \dots, x_n\rangle) \clubsuit^n |x_1, \dots, x_n\rangle$; $\sqrt{\texttt{Not}}^{(n)}(|x_1, \dots, x_n\rangle) \clubsuit^n |x_1, \dots, 1 - x_n\rangle$.
(10) For any $m, n \geq 1$, $\texttt{T}$ associates to the triplet $(m, n, 1)$ the gate $\texttt{T}^{(m,n,1)}$, defined on $\mathcal{Q}^{(m+n+1)}$. We put:

$$\texttt{And}(|\psi\rangle^{(m)}, |\varphi\rangle^{(n)}) := \texttt{T}^{(m,n,1)}(|\psi\rangle^{(m)} \otimes |\varphi\rangle^{(n)} \otimes |0\rangle);$$

$$\texttt{Or}(|\psi\rangle^{(m)}, |\varphi\rangle^{(n)}) := \texttt{Not}(\texttt{And}(\texttt{Not}(|\psi\rangle^{(m)}, \texttt{Not}(|\varphi\rangle^{(n)}))).$$

The following conditions hold:

(a) $\mathtt{T}^{(\mathtt{m},\mathtt{n},1)}(|\mathtt{x}_1,\ldots,\mathtt{x}_\mathtt{m},\mathtt{y}_1,\ldots,\mathtt{y}_\mathtt{n},\mathtt{z}\rangle \equiv |\mathtt{x}_1,\ldots,\mathtt{x}_\mathtt{m},\mathtt{y}_1,\ldots,\mathtt{y}_\mathtt{n},\mathtt{x}_\mathtt{m}\cdot\mathtt{y}_\mathtt{n}\boxplus\mathtt{z}\rangle$, where $\boxplus$ is the sum modulo 2.
(b) $\mathtt{And}(|\psi\rangle,|\varphi\rangle)\equiv\mathtt{And}(|\varphi\rangle,|\psi\rangle)$ (commutativity).
(c) $\mathtt{And}\{|\psi\rangle,\mathtt{And}(|\varphi\rangle,|\chi\rangle)\}\equiv\mathtt{And}\{\mathtt{And}(|\psi\rangle,|\varphi\rangle),|\chi\rangle\}$ (associativity).
(d) $\mathtt{And}\{|\psi\rangle,\mathtt{Or}(|\varphi\rangle,|\chi\rangle)\}\preceq\mathtt{Or}\{\mathtt{And}(|\psi\rangle,|\varphi\rangle),\mathtt{And}(|\psi\rangle,|\chi\rangle)\}$ (semidistributivity).
(e) $\sqrt{\mathtt{Not}}^{(1)}|1\rangle\preceq\sqrt{\mathtt{Not}}^{(m+n+1)}(\mathtt{T}^{(m,n,1)}(|\psi^{(m)}\rangle\otimes|\varphi^{(m)}\rangle\otimes|0\rangle))\preceq\sqrt{\mathtt{Not}}^{(1)}|0\rangle$.

One can easily show that the notion of *abstract quregister structure* represents a "good" abstraction from Hilbert-space quregisters. Consider the concrete structure

$$\langle\mathcal{Q},\clubsuit,\preceq,\mathtt{Not},\sqrt{\mathtt{Not}},\mathtt{T},|0\rangle,|1\rangle\rangle,$$

where:

- $\mathcal{Q}=\bigcup_{n\geq1}\mathcal{Q}(\otimes^n\mathbb{C}^2)$ is the set of all concrete quregisters;
- $\clubsuit^n$ is defined as follows.

(1) For any $|\psi\rangle^{(n)}=\sum_i c_i|x_{i_1},\ldots,x_{i_n}\rangle$,

$$|\psi\rangle^{(n)}\clubsuit^n|x_1,\ldots,x_n\rangle \quad \text{iff for some } c_i\neq0, \quad |x_1,\ldots,x_n\rangle=|x_{i_1},\ldots,x_{i_n}\rangle.$$

(2) $|\psi\rangle^{(n)}\clubsuit^n|\varphi\rangle^{(n)}$ iff there exists a register $|x_1,\ldots,x_n\rangle$ such that

$$|\psi\rangle^{(n)}\clubsuit^n|x_1,\ldots,x_n\rangle \quad \text{and} \quad |\varphi\rangle^{(n)}\clubsuit^n|x_1,\ldots,x_n\rangle.$$

- the relation $\preceq$, the gates $\mathtt{Not}$, $\sqrt{\mathtt{Not}}$, $\mathtt{T}$ and the two bits $|0\rangle$, $|1\rangle$ are defined according to the definitions given in Sect. 1.

This structure satisfies our definition of abstract quregister structure.

References

1. Dalla Chiara, M.L., Giuntini, R.: Quantum logics. In: Gabbay, G., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. VI, pp. 129–228. Kluwer, Dordrecht (2002)
2. Dalla Chiara, M.L., Giuntini, R., Greechie, R.: Reasoning in Quantum Theory. Kluwer, Dordrecht (2004)
3. Dalla Chiara, M.L., Giuntini, R., Leporini, R.: Quantum Computational Logics. A Survey. In: Hendricks, V., Malinowski, J. (eds.) Trends in Logic. 50 Years of Studia Logica, pp. 229–271. Kluwer, Dordrecht (2003)
4. Dalla Chiara, M.L., Giuntini, R., Leporini, R.: Logics from quantum computation. Int. J. Quantum Inf. **3**, 293–337 (2005)
5. Dalla Chiara, M.L., Giuntini, R., Leporini, R.: A holistic quantum computational semantics. Nat. Comput., 10.1007/s11047-006-9020-x, 1–20 (2006). ISSN: 1567-7818 (Print) 1572-9796 (Online)
6. Dalla Chiara, M.L., Giuntini, R., Gudder, S., Leporini, R.: Quantum computational semantics on Fock space. Int. J. Theor. Phys. **44**, 2219–2230 (2005)
7. Deutsch, D., Ekert, A., Lupacchini, R.: Machines, logic and quantum physics. Bull. Symb. Log. **3**, 265–283 (2000)
8. Gudder, S.: Quantum computational logic. Int. J. Theor. Phys. **42**, 39–47 (2003)